ON THE TENSORIAL CHARACTERISTICS OF FINITE DEFORMATIONS

(O TENZORNYKH KHARAKTERISTIKAKH KONECHNYKH DEFORMATSII)

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M. E. EGLIT
(Moscow)

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When finite deformations [strains] in elastic-plastic bodies are studied, the choice of the tensor which describes the total deformations becomes essential, as well as the determination of the separate tensors of the elastic and plastic deformations.

Independently of the choice of the tensor which describes the total deformations we shall subject the elastic and plastic deformation tensors to the following conditions:

1) They shall be introduced analogously to the total deformation tensor.

Indeed, the plastic deformation tensor is the total deformation tensor for processes which end in complete unloading. At that time, the entire deformed state can be assumed to be elastic if one does not refer it to the actual initial state but to the state which corresponds to complete unloading from the given state.

2) The elastic and plastic deformation tensors shall be introduced independently from one another.

Let, for instance, the deformations be described by the tensor ϵ:

$$\varepsilon = \varepsilon_{ij}\hat{\mathfrak{s}}^i\hat{\mathfrak{s}}^j$$

where

$$\varepsilon_{ij} = \frac{1}{2}(\hat{g}_{ij} - \mathring{g}_{ij}) = \frac{1}{2}(\nabla_i w_j + \nabla_j w_i - \nabla_i w^\alpha \nabla_j w_\alpha) \qquad (1)$$

wherein $\hat{\mathfrak{s}}^i$ are the unit vectors, fixed in the medium, forming the basis of the Lagrangean coordinate system ξ^i, g_{ij} are the components of the metric tensor **G** in the final (deformed) state of the material; $\overset{\circ}{\partial}{}^i$, $\overset{\circ}{g}_{ij}$ are the same quantities in the original state; w_i, w^i are the components of the displacement vector $\overline{\mathbf{w}} = w^i\,\hat{\partial}_i = w_i\hat{\partial}^i$.

Let us study, together with the initial state ($^\circ$) and the final state ($\hat{}$), the state ($*$), which may be obtained from the state ($\hat{}$) by the process of complete unloading. The components of the metric tensor **G** in the state ($*$) shall be denoted by $\overset{*}{g}_{ij}$, and the base vectors of the Lagrangean coordinate system by $\overset{*}{\partial}_i$.

The elastic ϵ^e and the plastic ϵ^p deformation tensors are usually expressed in terms of the following formulas:

$$\begin{aligned} \varepsilon^e &= (\varepsilon_{ij})^e\,\hat{\partial}^i\hat{\partial}^j, \qquad (\varepsilon_{ij})^e = \tfrac{1}{2}\,(\hat{g}_{ij} - \overset{*}{g}_{ij}),\\ \varepsilon^p &= (\varepsilon_{ij})^p\,\overset{*}{\partial}{}^i\,\overset{*}{\partial}{}^j, \qquad (\varepsilon_{ij})^p = \tfrac{1}{2}\,(\overset{*}{g}_{ij} - \overset{\circ}{g}_{ij}) \end{aligned} \tag{2}$$

Herein, the following equation holds for the covariant components of the tensors ϵ^e, ϵ^p and ϵ, which refer, respectively, to the bases $\hat{\partial}^i$, $\overset{*}{\partial}{}^i$ and $\hat{\partial}^i$:

$$(\varepsilon_{ij})^e + (\varepsilon_{ij})^p = \varepsilon_{ij} \tag{3}$$

This equation is independent of the choice of the Lagrangean coordinate system ξ^i and transforms tensorially from the system ξ^i to another Lagrangean system η^i. It cannot, however, be written as a tensor relation between ϵ, ϵ^e and ϵ^p and is not satisfied for contravariant and mixed components of these tensors. The equation, also, does not hold for components computed in some Eulerian reference system. For instance, for the mixed components $(\epsilon_{i\cdot}^{\cdot j})^e$, $(\epsilon_{i\cdot}^{\cdot j})^p$ and $\epsilon_{i\cdot}^{\cdot j}$ we have:

$$[\delta_{i\cdot}^{\cdot\alpha} - 2\,(\varepsilon_{i\cdot}^{\cdot\alpha})^p]\,[\delta_{\alpha\cdot}^{\cdot j} - 2\,(\varepsilon_{\alpha\cdot}^{\cdot j})^e] = \delta_{i\cdot}^{\cdot j} - 2\varepsilon_{i\cdot}^{\cdot j} \tag{4}$$

i.e.

$$\varepsilon_{i\cdot}^{\cdot j} = (\varepsilon_{i\cdot}^{\cdot j})^e + (\varepsilon_{i\cdot}^{\cdot j})^p - 2\,(\varepsilon_{i\cdot}^{\cdot\alpha})^p\,(\varepsilon_{\alpha\cdot}^{\cdot j})^e \tag{5}$$

Thus, if the deformations are described by the quantities $\epsilon_{i\cdot}^{\cdot j}$, $(\epsilon_{i\cdot}^{\cdot j})^e$ and $(\epsilon_{i\cdot}^{\cdot j})^p$, then, for finite deformations, the elastic deformations are not equal to the difference between the total and the plastic deformations. This effect is also apparent when the elastic deformations are small, as long as the plastic deformations are finite.

Various functions of the tensor ϵ can be employed as the deformation characteristic. If the total deformations are described by the tensor $\mathbf{T} = j(\epsilon)$, then it follows from conditions (1) and (2) that

$$\mathbf{T}^e = f(\boldsymbol{\varepsilon}^e), \qquad \mathbf{T}^p = f(\boldsymbol{\varepsilon}^p)$$

One can, for instance, use the "strain" tensors **E** and "true deformation" tensors **H**, which are given by the formulas

$$\mathbf{E} = \mathbf{G} - \sqrt{\mathbf{G} - 2\boldsymbol{\varepsilon}} = E_{ij}\hat{\partial}^i\hat{\partial}^j, \qquad \mathbf{H} = -\tfrac{1}{2}\ln(\mathbf{G} - 2\boldsymbol{\varepsilon}) = h_{ij}\hat{\partial}^i\hat{\partial}^j \tag{6}$$

The principal values of these tensors are especially simply related to the initial l_i° and the final l_i length of the element lying along the ith principal axis of the deformation tensor

$$E_i = \frac{l_i - l_i^\circ}{l_i}, \qquad h_i = \ln\frac{l_i}{l_i^\circ}$$

The values E_i and h_i are often used in experimental work. The elastic and plastic deformations should then be determined by the formulas

$$\mathbf{E}^e = \mathbf{G} - \sqrt{\mathbf{G} - 2\boldsymbol{\varepsilon}^e}, \quad \mathbf{E}^p = \mathbf{G} - \sqrt{\mathbf{G} - 2\boldsymbol{\varepsilon}^p}, \quad E_i^e = \frac{l_i - l_i^p}{l_i}, \quad E_i^p = \frac{l_i^p - l_i^\circ}{l_i^p} \tag{7}$$

and also

$$\mathbf{H}^e = -\tfrac{1}{2}\ln(\mathbf{G} - 2\boldsymbol{\varepsilon}^e), \quad \mathbf{H}^p = -\tfrac{1}{2}\ln(\mathbf{G} - 2\boldsymbol{\varepsilon}^p), \quad h_i^e = \ln\frac{l_i}{l_i^p}, \quad h_i^p = \ln\frac{l_i^p}{l_i^\circ} \tag{8}$$

The tensors **E** and **H** are now linear functions of the deformation tensor $\boldsymbol{\epsilon}$. In different problems it may be convenient to use also other functions of $\boldsymbol{\epsilon}$, for instance, $\boldsymbol{\Theta}$:

$$\boldsymbol{\Theta} = \boldsymbol{\varepsilon}(\mathbf{G} - 2\boldsymbol{\varepsilon})^{-1} = \tfrac{1}{2}[(\mathbf{G} - 2\boldsymbol{\varepsilon})^{-1} - \mathbf{G}] \tag{9}$$

The contravariant components of the tensor $\boldsymbol{\Theta}$ with respect to the basis $\hat{\partial}_i$ can be simply expressed by means of the contravariant components of the metric tensor **G** with respect to the bases $\hat{\partial}_i$ and $\overset{\circ}{\partial}_i$:

$$\Theta = \theta^{ij}\hat{\partial}_i\hat{\partial}_j, \qquad \theta^{ij} = \tfrac{1}{2}(\overset{\circ}{g}{}^{ij} - \hat{g}^{ij})$$

Let us assume that

$$\boldsymbol{\Theta}^e = \boldsymbol{\varepsilon}^e(\mathbf{G} - 2\boldsymbol{\varepsilon}^e)^{-1} = (\theta^{ij})^e\hat{\partial}_i\hat{\partial}_j, \qquad (\theta^{ij})^e = \tfrac{1}{2}(\overset{*}{g}{}^{ij} - \hat{g}^{ij})$$

$$\boldsymbol{\Theta}^p = \boldsymbol{\varepsilon}^p(\mathbf{G} - 2\boldsymbol{\varepsilon}^p)^{-1} = (\theta^{ij})^p\overset{*}{\partial}_i\overset{*}{\partial}_j, \qquad (\theta^{ij})^p = \tfrac{1}{2}(\overset{\circ}{g}{}^{ij} - \overset{*}{g}{}^{ij})$$

Thus, we have

$$\theta^{ij} = (\theta^{ij})^e + (\theta^{ij})^p \tag{10}$$

Analogous equations for the corresponding covariant and mixed components are not satisfied. It is evident from (4) and (8) that if the principal axes of the elastic and plastic deformation tensors coincide, i.e. the matrices

$$\|\delta_{i\cdot}^{\cdot j}-2\,(\varepsilon_{i\cdot}^{\cdot j})^{e}\|, \qquad \|\delta_{i\cdot}^{\cdot j}-2\,(\varepsilon_{i\cdot}^{\cdot j})^{p}\|$$

are commutative, then the following equation holds:

$$(h_{i\cdot}^{\cdot j})^{e}+(h_{i\cdot}^{\cdot j})^{p}=h_{i\cdot}^{\cdot j} \tag{11}$$

which relates the mixed components of the tensors:

$$\mathbf{H}=h_{i\cdot}^{\cdot j}\hat{\partial}^{i}\hat{\partial}_{j}, \qquad \mathbf{H}^{e}=(h_{i\cdot}^{\cdot j})^{e}\hat{\partial}^{i}\hat{\partial}_{j}, \qquad \mathbf{H}^{p}=(h_{i\cdot}^{\cdot j})^{p}\overset{*}{\partial}{}^{i}\overset{*}{\partial}_{j}$$

This will always be the case when in the process of deformation the principal axes of deformation of the element do not rotate. Consequently, when simple loadings or some forms of complex loadings are described, one can assume, when using the tensor **H**, that the sum of the elastic and the plastic deformations is equal to the total deformation. One should not, however, generalize this to the case of arbitrary deformations. Relations (3) and (10), on the other hand, are always correct.

The use of the quantities ϵ_{ij} and θ^{ij} is also helpful in another context. The deformation tensors should be so chosen that the stress-deformation relations would have a most convenient form. The characteristics of these relations are largely determined by thermodynamics; in particular, the expression for the elemental work of the internal forces has a greater meaning. It is well-known that the elemental work of the internal forces per unit mass is equal to

$$dA^{(i)}=-\frac{1}{\rho}\,p^{\alpha\beta}d\varepsilon_{\alpha\beta} \tag{12}$$

Here ρ is the density of the material $\mathbf{P}=p^{\alpha\beta}\hat{\partial}_{\alpha}\hat{\partial}_{\beta}=p_{\alpha\beta}\hat{\partial}^{\alpha}\hat{\partial}^{\beta}$ is the stress tensor, $\epsilon_{\alpha\beta}$ are the covariant components of the tensor ϵ with the basis $\hat{\partial}^{\alpha}$.

The work $dA^{(i)}$ can be expressed just as simply by the introduction of $\theta^{\alpha\beta}$; then we have

$$dA^{(i)}=-\frac{1}{\rho}\,p_{\alpha\beta}\,d\theta^{\alpha\beta} \tag{13}$$

Indeed,

$$p^{ij}d\varepsilon_{ij}=p_{\alpha\beta}\hat{g}^{\alpha i}\hat{g}^{\beta j}d\varepsilon_{ij}=\frac{1}{2}\,p_{\alpha\beta}\hat{g}^{\alpha i}\hat{g}^{\beta j}d\hat{g}_{ij}=-\frac{1}{2}\,p_{\alpha\beta}d\hat{g}^{\alpha\beta}=p_{\alpha\beta}\,d\theta^{\alpha\beta}$$

If the material is such that the principal axes of the stress tensor coincide with the principal axes of the deformation tensor (as, for instance, in the isotropic elastic body), then

$$dA^{(i)}=-\frac{1}{\rho}\,p_{\cdot\beta}^{\alpha\cdot}\,dh_{\cdot\alpha}^{\beta\cdot} \tag{14}$$

In the converse case this is not true, and one can show than that a tensor $\mathbf{L} = l_{i\cdot}^{\cdot j}\hat{\partial}^i\hat{\partial}_j = f(\mathbf{e})$ does not exist which would satisfy the equation

$$dA^{(i)} = -\frac{1}{\rho}p^{\alpha\cdot}_{\cdot\beta}\,dl^{\beta\cdot}_{\cdot\alpha}$$

The representation of the work of the internal forces in the form (12) (13) or (14) leads to the fact that the corresponding components of the tensor $(1/\rho)\mathbf{P}$ possess a potential and are derivatives of the free energy with respect to the elastic deformations.

Thus, of all the above-studied deformation characteristics, the quantities $\epsilon_{\alpha\beta}$, $\theta^{\alpha\beta}$ and, to a limited extent, $h_{\alpha}^{\cdot\beta}$ possess two convenient properties:

1. The laws of elasticity for an arbitrary elastic-plastic medium are represented in terms of these variables in most symmetric form as

$$\frac{1}{\rho}p^{\alpha\beta} = \frac{\partial F}{\partial(\varepsilon_{\alpha\beta})^e}, \quad \text{or} \quad \frac{1}{\rho}p_{\alpha\beta} = \frac{\partial F}{\partial(\theta^{\alpha\beta})^e}, \quad \text{or} \quad \frac{1}{\rho}p^{\alpha\cdot}_{\cdot\beta} = \frac{\partial F}{\partial(h_{\alpha\cdot}^{\cdot\beta})^e} \tag{15}$$

(The last relation holds when (14) is correct).

2. The elastic deformations are equal to the difference between the total and the plastic deformations, and thus it is convenient to represent graphs in the form "stress versus total deformation." Actually, the unloading lines on these diagrams give a graph of the laws (15) shifted with respect to the origin by the amount of the plastic deformations.

In some cases, however, the values $\epsilon_{\alpha\beta}$ and $\theta^{\alpha\beta}$ are not the most convenient characteristics of the deformations.

For instance, let us analyze the problem of the influence of the plastic deformation upon the elastic properties of the material. We shall show by means of a simple example that the correct choice of the deformation characteristic is of essential importance.

Let us study a simple stretching of a cylindrical specimen which has an initial length l°. Assume that l is its final length, l^* is the corresponding residual length, and the deformation of the sample is characterized by a strain $E = (l - l^\circ)\, l^\circ$. Then (according to conditions (1) and (2))

$$\overset{\circ}{E}{}^p = \frac{l^* - l^\circ}{l^\circ}, \qquad \overset{*}{E}{}^e = \frac{l - l^*}{l^*}$$

At the same time, let us study also three other characteristics of the elastic deformation

$$\widetilde{E}^e = \hat{E} - \mathring{E}^p = \mathring{E}^e \frac{l^*}{l^\circ}$$

$$(\varepsilon_{11})^e = \frac{1}{2}[(1 + \mathring{E}^e)^2 - 1](1 + \mathring{E}^p)^2$$

$$(\theta^{11})^e = \frac{1}{2}[1 - (1 + \mathring{E}^e)^{-2}](1 + \mathring{E}^p)^{-2}$$

The last two formulas give the expressions $(\epsilon_{11})^e$ and $(\theta_{11})^e$ in terms of the relative elongation with the condition that the axes of the Lagrangean coordinate system ξ^i coincide with the distances of the principal deformation axes, and the axis ξ^1 coincides with the axis of the specimen.

Let us assume that the elastic relative elongations satisfy one and the same linear Hooke's law independently of the magnitude of the plastic deformations, i.e., that $\mathring{E}^e = k\sigma$.

Then, when using the values $\widetilde{E}^e$, $(\epsilon_{11})^e$, $(\theta^{11})^e$, the laws of elasticity for this medium will become

$$\widetilde{E}^e = k \frac{l^*}{l^\circ} \sigma = k(1 + \mathring{E}^p)\sigma$$

$$(\varepsilon_{11})^e = \frac{1}{2}[(1 + k\sigma)^2 - 1](1 + \mathring{E}^p)^2$$

$$(\theta^{11})^e = \frac{1}{2}[1 - (1 + k\sigma)^{-2}](1 + \mathring{E}^p)^{-2}$$

i.e. they clearly contain the characteristics of the plastic deformation.

Thus, the problem of the dependence of the elastic laws on the plastic deformation is not only connected with the physical processes that take place in the material but also with the choice of the deformation characteristics.

Let us introduce the following definition.

The elastic-plastic medium is a medium whose elastic properties do not change with plastic deformation, if its free energy F can be represented in the form

$$F(\mathring{g}_{ij}, (\varepsilon_{ij})^e, (\varepsilon_{ij})^p, \chi_s, T) = F_1(\overset{*}{g}_{ij}, (\varepsilon_{ij})^e, T) + F_2(\mathring{g}_{ij}, (\varepsilon_{ij})^p, \chi_s, T) \qquad (16)$$

$$\overset{*}{g}_{ij} = \mathring{g}_{ij} + 2(\varepsilon_{ij})^p \qquad (\varepsilon_{ij})^e = \frac{1}{2}(\hat{g}_{ij} - \overset{*}{g}_{ij})$$

Here $\overset{*}{g}_{ij}$ are the components of the metric tensor **G** in the state (*), which plays the role of the initial state during repeated loading; the parameters χ_s are the characteristics of the plastic state; they depend on the path of the plastic deformation.

For such a body, the laws of elasticity, written in the usual form,

$$\frac{1}{\rho}\, p^{\alpha\beta} = \frac{\partial F_1}{\partial\, (\varepsilon_{\alpha\beta})^e}$$

contain plastic-deformation characteristics and change during the plastic deformation.

The invariants of the tensor $(1/\rho)\mathrm{P}$, however, depend in this case only on the invariants of the elastic deformation tensor ϵ^e and the temperature T; the laws of elasticity written in terms of mixed components of the corresponding tensors do not depend on the plastic deformations:

$$F_1 = F_1\,((\overset{*}{\varepsilon}{}_{\alpha\cdot}^{\cdot\beta})^e,\ T), \qquad \frac{1}{\rho}\, \overset{*}{p}{}^{\alpha\cdot}_{\cdot\beta} = \frac{\partial F_1}{\partial\, (\overset{*}{\varepsilon}{}_{\alpha\cdot}^{\cdot\beta})^e} \tag{17}$$

$(\overset{*}{\varepsilon}{}_{\alpha\cdot}^{\cdot\beta})^e$ and $\overset{*}{p}{}^{\alpha\cdot}_{\cdot\beta}$ are related to $(\epsilon_{\alpha\cdot}^{\cdot\beta})^e$ and $p^{\alpha\cdot}_{\cdot\beta}$ by the formulas

$$(\overset{*}{\varepsilon}{}_{\alpha\cdot}^{\cdot\beta})^e = (\varepsilon_{\alpha\cdot}^{\cdot\gamma})^e\,[\delta_{\gamma\cdot}^{\cdot\beta} + 2(\varepsilon_{\gamma\cdot}^{\cdot\beta})^e], \qquad \overset{*}{p}{}^{\alpha\cdot}_{\cdot\beta} = p^{\alpha\cdot}_{\cdot\gamma}\,[\delta^{\gamma\cdot}_{\cdot\beta} - 2\,(\varepsilon^{\gamma\cdot}_{\cdot\beta})^e]$$

Thus, the influence of plasticity upon the elastic properties of the body is more conveniently studied if one constructs unloading lines in the plane of the variables $(1/\rho)p^{\alpha}{}_{\beta}$, $(\epsilon_{\alpha}{}^{\beta})^e$. Since

$$(\varepsilon_{\alpha\cdot}^{\cdot\beta})^e \neq \varepsilon_{\alpha\cdot}^{\cdot\beta} - (\varepsilon_{\alpha\cdot}^{\cdot\beta})^p$$

the unloading lines in the $(1/\rho)p^{\alpha\cdot}_{\cdot\beta}$, $\epsilon_{\alpha\cdot}^{\cdot\beta}$ diagrams change their form during plastic deformation for bodies with the elastic law (17).

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Translated by M.I.Y.